

CONSEQUENCES OF THE FUNDAMENTAL CONJECTURE FOR THE MOTION ON THE SIEGEL-JACOBI DISK

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ABSTRACT. We find the homogenous Kähler diffeomorphism FC which expresses the Kähler two-form on the Siegel-Jacobi domain $\mathcal{D}_1^J = \mathbb{C} \times \mathcal{D}_1$ as the sum of the Kähler two-form on \mathbb{C} and the one on the Siegel ball \mathcal{D}_1 . The classical motion and quantum evolution on \mathcal{D}_1^J determined by a linear Hamiltonian in the generators of the Jacobi group $G_1^J = H_1 \rtimes \mathrm{SU}(1, 1)$ is described by a Riccati equation on \mathcal{D}_1 and a linear first order differential equation in $z \in \mathbb{C}$, where H_1 denotes the 3-dimensional Heisenberg group. When the transformation FC is applied, the first order differential equation for the variable $z \in \mathbb{C}$ decouples of the motion on the Siegel disk. Similar considerations are presented for the Siegel-Jacobi space $\mathcal{X}_1^J = \mathbb{C} \times \mathcal{X}_1$, where \mathcal{X}_1 denotes the Siegel upper half plane.

1. INTRODUCTION

The Jacobi group [18] is the semidirect product $G_n^J = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$, where H_n denotes the real $(2n + 1)$ -dimensional Heisenberg group. Several generalizations are known [46], [30]. The homogenous Kähler Siegel-Jacobi domains $\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$, where \mathcal{D}_n is the Siegel ball, are nonsymmetric domains associated to the nonreductive Jacobi groups by the generalized Harish-Chandra embedding [39], [30], [50]-[52]. The holomorphic irreducible unitary representations of the Jacobi groups based on Siegel-Jacobi domains have been constructed [12, 13, 45, 46, 8].

Some coherent state systems based on Siegel-Jacobi domains have been investigated in the framework of quantum mechanics, geometric quantization, dequantization, quantum optics, nuclear structure, and signal processing [29, 38, 42, 5, 6, 7]. The Jacobi group was investigated by physicists under other names as *Hagen* [21], *Schrödinger* [34], or *Weyl-symplectic* group [49]. The Jacobi group is responsible for the *squeezed states* [28, 44, 31, 53, 22] in quantum optics [32, 1, 43, 17].

In the papers [5, 6] the Jacobi group is studied in connection with the group-theoretic approach to coherent states [37]. We have attached to the Jacobi group G_n^J coherent states based on Siegel-Jacobi disk \mathcal{D}_n^J [6]. In this paper we consider the case of the Jacobi group G_1^J [5]. We have determined the G_n^J -invariant Kähler two-form ω_n on \mathcal{D}_n^J [6] from the Kähler potential, and, with the partial Cayley transform, we have determined the Kähler two-form ω'_n on the Siegel-Jacobi upper-half plane $\mathcal{X}_n^J = \mathbb{C}^n \times \mathcal{X}_n$, where \mathcal{X}_n is the Siegel upper half-plane. The Kähler two form ω'_1 on $\mathcal{X}_1^J = \mathbb{C} \times \mathcal{X}_1$ was firstly investigated by Kähler himself [26] and Berndt [11], while ω_n and ω'_n have been

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investigated also by Yang [51],[52]. ω_1 is the sum of two terms, one $\omega_{\mathcal{D}_1}$ describing the the Kähler two form on \mathcal{D}_1 , the other one is $(1 - w\bar{w})^{-1}A \wedge \bar{A}$, where $A = dz + \bar{\eta}dw$, and $\eta = (1 - w\bar{w})^{-1}(z + \bar{z}w)$, $z \in \mathbb{C}, w \in \mathcal{D}_1$ [5]. Let us denote by FC the change of variables $FC : \mathcal{D}_1^J = \mathbb{C} \times \mathcal{D}_1 \ni (z, w) \rightarrow (\eta, w)$. It is find out that if we express ω_1 in the coordinates (η, w) , then the Kähler two-form becomes $\omega_0 := FC^*(\omega_1) = \omega_{\mathcal{D}_1} + \omega_{\mathbb{C}}$, invariant to the action of G_1^J on the product manifold $\mathbb{C} \times \mathcal{D}_1$. We put this change of variables in connection with the celebrated fundamental conjecture of Gindikin-Vinberg [47],[16] on the homogeneous Kähler Siegel-Jacobi disk. Similar considerations are presented for the homogenous Kähler Siegel-Jacobi space \mathcal{X}_1^J .

Dequantization in the Berezin approach [9],[10] of a dynamical system problem with Lie group of symmetry G on a Hilbert space \mathcal{H} in the simple case of linear Hamiltonian was considered in [3],[4]. Linear Hamiltonians in generators of the Jacobi group appear in many physical problems of quantum mechanics, as in the case of the quantum oscillator acted on by a variable external force [19], [40], [23]. The same problem was considered in the group-theoretic approach in [35, 36, 37], where linear Hamiltonians in the generators of $SU(1, 1)$ or $Isp(2)$ are considered. A similar treatment has been used in the case of quantum dynamics of trapped ions [20].

The paper is laid out as follows. §2, devoted to G_1^J , starts in §2.1 with the Jacobi algebra \mathfrak{g}_1^J and Perelomov's coherent states defined on \mathcal{D}_1^J . Several facts concerning a holomorphic representation of \mathfrak{g}_1^J as first order differential operators with polynomial coefficients on \mathcal{D}_1^J are summarized in Lemma 1, which is essential for §4. Remark 1 is new. In §2.2 we recall some facts referring to the real Jacobi group $G_1^J(\mathbb{R})$, firstly considered by Kähler and Berndt [24, 25, 26, 11, 13]. Proposition 1 and Remark 2 are extracted from [5]. In §3 we find out the homogenous Kähler isomorphism $FC : \mathcal{D}_1^J \rightarrow \mathbb{C} \times \mathcal{D}_1$ ($FC_1 : \mathcal{X}_1^J \rightarrow \mathbb{C} \times \mathcal{X}_1$, respectively). In §4 we study classical and quantum motion determined by linear Hamiltonians in generators of the Jacobi group G_1^J on Siegel-Jacobi disk and upper half plane. We calculate the Berry phase and the dynamical phase [41] on \mathcal{D}_1^J determined by a Hamiltonian linear in the generators of the Jacobi group. The simple example of an autonomous system is explicitly solved. The main results of the paper are stated in Proposition 2, Corollary 1, and Proposition 4.

2. THE JACOBI GROUP G_1^J

2.1. The Jacobi algebra \mathfrak{g}_1^J and Perelomov's coherent states. We consider a realization of the 3-dimensional Heisenberg Lie algebra

$$(2.1) \quad \mathfrak{h}_1 \equiv \langle is1 + xa^\dagger - \bar{x}a \rangle_{s \in \mathbb{R}, x \in \mathbb{C}},$$

where a^\dagger (a) are the boson creation (respectively, annihilation) operators, $[a, a^\dagger] = 1$.

The Jacobi algebra is defined as the the semi-direct sum $\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1, 1)$, where

$$(2.2) \quad \mathfrak{su}(1, 1) = \langle 2i\theta K_0 + yK_+ - \bar{y}K_- \rangle_{\theta \in \mathbb{R}, y \in \mathbb{C}},$$

$$(2.3) \quad [K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0;$$

$$(2.4a) \quad [a, K_+] = a^\dagger, [K_-, a^\dagger] = a, [K_+, a^\dagger] = [K_-, a] = 0,$$

$$(2.4b) \quad [K_0, a^\dagger] = \frac{1}{2}a^\dagger, [K_0, a] = -\frac{1}{2}a.$$

Let us suppose that we know the derived representation $d\pi$ of the Lie algebra \mathfrak{g}_1^J of the Jacobi group G_1^J . For a Lie algebra \mathfrak{g} , if $X \in \mathfrak{g}$, we denote $\mathbf{X} = d\pi(X)$. We impose to the cyclic vector e_0 to verify simultaneously the conditions

$$(2.5) \quad \mathbf{a}e_0 = 0, \mathbf{K}_-e_0 = 0, \mathbf{K}_0e_0 = ke_0; \quad k > 0, 2k = 2, 3, \dots,$$

and we have considered in the last relation in (2.5) the positive discrete series representations D_k^+ of $SU(1, 1)$ [2].

Perelomov's coherent state vectors associated to the group G_1^J with Lie algebra the Jacobi algebra \mathfrak{g}_1^J , based on Siegel-Jacobi disk $\mathfrak{D}_1^J = H_1/\mathbb{R} \times SU(1, 1)/U(1) = \mathbb{C} \times \mathcal{D}_1$, are defined as

$$(2.6) \quad e_{z,w} := e^{z\mathbf{a}^\dagger + w\mathbf{K}_+}e_0, \quad z, w \in \mathbb{C}, \quad |w| < 1.$$

The formulas below are obtained [5] using the relation $\text{Ad}(\exp X) = \exp(\text{ad}_X)$

$$(2.7a) \quad \mathbf{a}^+e_{z,w} = \frac{\partial}{\partial z}e_{z,w}; \quad \mathbf{a} = (z + w\frac{\partial}{\partial z})e_{z,w};$$

$$(2.7b) \quad \mathbb{K}_+e_{z,w} = \frac{\partial}{\partial w}e_{z,w}; \quad \mathbb{K}_0 = (k + \frac{1}{2}z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w})e_{z,w};$$

$$(2.7c) \quad \mathbb{K}_-e_{z,w} = (\frac{1}{2}z^2 + 2kw + zw\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w})e_{z,w}.$$

With (2.7), the general scheme [3, 4] associates to elements of the Lie algebra \mathfrak{g} , first order holomorphic differential operators with polynomial coefficients $X \in \mathfrak{g} \rightarrow \mathbb{X}$:

Lemma 1. *The differential action of the generators of the Jacobi algebra (2.4) is given by the formulas:*

$$(2.8a) \quad \mathbf{a} = \frac{\partial}{\partial z}; \quad \mathbf{a}^+ = z + w\frac{\partial}{\partial z}, \quad z, w \in \mathbb{C}, \quad |w| < 1;$$

$$(2.8b) \quad \mathbb{K}_- = \frac{\partial}{\partial w}; \quad \mathbb{K}_0 = k + \frac{1}{2}z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w};$$

$$(2.8c) \quad \mathbb{K}_+ = \frac{1}{2}z^2 + 2kw + zw\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}.$$

Acting on $e_{z,w}$, the differential operators (2.7) are not independent:

Remark 1. *Perelomov's coherent state vector (2.6) verifies the system of differential equations $Xe_{z,w} = 0, Ye_{z,w} = 0$, where*

$$(2.9) \quad X = k\mathbf{a} + (\frac{z^2}{2} - kw)\mathbf{a}^+ + wk\mathbb{K}_+ - z\mathbb{K}_0,$$

$$(2.10) \quad Y = \frac{1}{2}z^3\mathbf{a}^+ + (2kw + z^2)w\mathbb{K}_+ - (4kw + z^2)\mathbb{K}_0 + 2k\mathbb{K}_-.$$

Proof. We want to determine the constants $A - E$ such that

$$(A\mathbf{a} + B\mathbf{a}^\dagger + C\mathbb{K}_+ + D\mathbb{K}_0 + E\mathbb{K}_-)e_{z,w} = 0.$$

We use the relations (2.7), and equate with 0 the coefficients of *constant*, $\frac{\partial}{\partial z}$, and $\frac{\partial}{\partial w}$. We express four of the constants $A - E$ as function of other two. Choosing A and E as independent, we get the expressions given in (2.9). \blacksquare

We consider the displacement operator

$$(2.11) \quad D(\alpha) = \exp(\alpha\mathbf{a}^\dagger - \bar{\alpha}\mathbf{a}) = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha\mathbf{a}^\dagger) \exp(-\bar{\alpha}\mathbf{a}),$$

and let us denote by S the unitary squeezed operator – the D_+^k representation of the group $SU(1, 1)$. We introduce the notation $\underline{S}(z) = S(w)$, $z, w \in \mathbb{C}$, $|w| < 1$, where

$$(2.12a) \quad \underline{S}(z) = \exp(z\mathbf{K}_+ - \bar{z}\mathbf{K}_-) = \exp(w\mathbf{K}_+) \exp(\eta\mathbf{K}_0) \exp(-\bar{w}\mathbf{K}_-),$$

$$(2.12b) \quad w = \frac{z}{|z|} \tanh(|z|), \quad \eta = \log(1 - w\bar{w}).$$

We introduce also the normalized (*squeezed*) CS vector $\Psi_{\alpha,w} := D(\alpha)S(w)e_0$ [44].

We introduce the auxiliary operators [5]:

$$(2.13) \quad \mathbf{K}_+ = \frac{1}{2}(\mathbf{a}^\dagger)^2 + \mathbf{K}'_+, \quad \mathbf{K}_- = \frac{1}{2}(\mathbf{a}^\dagger)^2 + \mathbf{K}'_-, \quad \mathbf{K}_0 = \frac{1}{2}(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}) + \mathbf{K}'_0,$$

which have the properties

$$(2.14) \quad \mathbf{K}'_- e_0 = 0, \quad \mathbf{K}'_0 e_0 = k' e_0; \quad k = k' + \frac{1}{4};$$

$$(2.15) \quad [\mathbf{K}'_\sigma, \mathbf{a}] = [\mathbf{K}'_\sigma, \mathbf{a}^\dagger] = 0, \quad \sigma = \pm, 0, \quad [\mathbf{K}'_0, \mathbf{K}'_\pm] = \pm \mathbf{K}'_\pm; \quad [\mathbf{K}'_-, \mathbf{K}'_+] = 2\mathbf{K}'_0.$$

We recall the orthonormal system of coherent states associated to the group $SU(1, 1)$:

$$(2.16) \quad e_{k,k+m} := a_{km}(\mathbf{K}_+)^m e_{k,k}; \quad a_{km}^2 = \frac{\Gamma(2k)}{m! \Gamma(m+2k)},$$

and to the Heisenberg-Weyl group

$$(2.17) \quad \varphi_n = (n!)^{-\frac{1}{2}} (a^+)^n \varphi_0; \quad \langle \varphi_{n'}, \varphi_n \rangle = \delta_{nn'}.$$

We write down the vector e_0 in (2.5) as

$$(2.18) \quad e_0 = e_0^H \otimes e_0^{K'}, \quad \text{where } \varphi_0 \equiv e_0^H; \quad e_0^{K'} \equiv e_{k',k'}.$$

Proposition 1. The kernel $K(z, w; \bar{z}', \bar{w}') := (e_{\bar{z}, \bar{w}}, e_{\bar{z}', \bar{w}'}) : \mathcal{D}_1^J \times \bar{\mathcal{D}}_1^J \rightarrow \mathbb{C}$ is:

$$(2.19) \quad K(z, w; \bar{z}', \bar{w}') = (1 - w\bar{w}')^{-2k} \exp(F(z, w; \bar{z}', \bar{w}')); \quad F = \frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2w}{2(1 - w\bar{w}')},$$

$$(2.20) \quad K = K(\bar{z}, \bar{w}, z, w) = (1 - w\bar{w})^{-2k} \exp \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2w}{2(1 - w\bar{w})}, \quad z, w \in \mathbb{C}, \quad |w| < 1.$$

The normalized squeezed state vector and the un-normalized Perelomov's coherent state vector are related by the relation

$$(2.21) \quad \Psi_{\alpha,w} = (1 - w\bar{w})^k \exp(-\frac{\bar{\alpha}}{2}z) e_{z,w}, \quad z = \alpha - w\bar{\alpha}.$$

The composition law in the Jacobi group $G_1^J := HW \rtimes SU(1, 1)$ is

$$(2.22) \quad (g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)),$$

where $g_i, i = 1, 2$ are of the form

$$(2.23) \quad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, |a|^2 - |b|^2 = 1,$$

$g \cdot \alpha := \alpha_g$ is given by $\alpha_g = a\alpha + b\bar{\alpha}$, $g^{-1} \cdot \alpha = \bar{a}\alpha - b\bar{\alpha}$.

Let $(g, \alpha) \in G_1^J$ and let $(z, w) \in \mathcal{D}_1^J := \mathbb{C} \times \mathcal{D}_1$. The action of the group G_1^J on the manifold \mathcal{D}_1^J is given by

$$(2.24) \quad z_1 = \frac{\alpha - \bar{\alpha}w + z}{\bar{b}w + \bar{a}}; \quad w_1 = g \cdot w = \frac{aw + b}{\bar{b}w + \bar{a}}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1, 1).$$

The scalar product of functions from the space \mathcal{F}_K corresponding to the kernel defined by (2.19) on the manifold \mathcal{D}_1^J , with G_1^J -invariant measure $d\nu$, is $(f_\psi(z, w) = (e_{\bar{z}, w}, \psi)_{\mathcal{H}})$:

$$(2.25) \quad (\phi, \psi) = \Lambda_1 \int_{z \in \mathbb{C}; |w| < 1} \bar{f}_\phi(z, w) f_\psi(z, w) (1 - w\bar{w})^{2k} \exp\left(-\frac{|z|^2}{1 - w\bar{w}}\right) \exp\left(-\frac{z^2 \bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})}\right) d\nu,$$

$$(2.26) \quad d\nu = \frac{d\Re w d\Im w}{(1 - w\bar{w})^3} d\Re z d\Im z, \quad \Lambda_1 = \frac{4k - 3}{2\pi^2}.$$

The Kähler potential $f := \log K$ is

$$(2.27) \quad f = \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})} - 2k \log(1 - w\bar{w}),$$

and the Kähler two-form ω_1 on \mathcal{D}_1^J , G_1^J -invariant to the action (2.24), is

$$(2.28) \quad -i\omega_1 = \frac{2k}{(1 - w\bar{w})^2} dw \wedge d\bar{w} + \frac{A \wedge \bar{A}}{1 - w\bar{w}}, \quad A = dz + \bar{\eta}dw, \quad \eta = \frac{z + \bar{z}w}{1 - w\bar{w}}.$$

2.2. The real Jacobi group. Let us now recall that $C^{-1}\mathrm{SL}_2(\mathbb{R})C = \mathrm{SU}(1, 1)$, where

$$(2.29) \quad C = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}; \quad C^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

If $M \in \mathrm{SL}_2(\mathbb{R})$ is the matrix

$$(2.30) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,$$

then

$$(2.31) \quad M_* = C^{-1}MC = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1.$$

The map (2.29) induces a transformation of the bounded domain \mathcal{D}_1 into the upper half plane \mathcal{X}_1 and

$$(2.32) \quad w = C^{-1}(v) = \frac{v - i}{v + i} \in \mathcal{D}_1; \quad v = Cw = i \frac{1 + w}{1 - w}.$$

Kähler and Berndt have investigated the Jacobi group $G_1^J(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \rtimes \mathbb{R}^2$ acting on the Siegel-Jacobi upper half plane $\mathcal{X}_1^J := \mathcal{X}_1 \times \mathbb{C}$ [11, 12, 13, 24, 25, 26, 27], where

\mathcal{X}_1 is the Siegel upper half plane $\mathcal{X}_1 := \{v \in \mathbb{C} | \Im(v) > 0\}$. It is easy to proof (see also ([5])) the following

Remark 2. *The action $C^{-1}G_1^J(\mathbb{R})C$ descends on the basis to the biholomorphic map: $\check{C}^{-1} : \mathcal{X}_1^J := \mathcal{X}_1 \times \mathbb{C} \rightarrow \mathcal{D}_1^J := \mathcal{D}_1 \times \mathbb{C}$:*

$$(2.33) \quad w = \frac{v - i}{v + i}; \quad z = \frac{2iu}{v + i}, \quad w \in \mathcal{D}_1, \quad v \in \mathcal{X}_1, \quad z \in \mathbb{C}.$$

Under the partial Cayley transform (2.33), the Kähler two-form ω_1 (2.28) becomes

$$(2.34) \quad -i \omega'_1 = -\frac{2k}{(\bar{v} - v)^2} dv \wedge d\bar{v} + \frac{2}{i(\bar{v} - v)} B \wedge \bar{B}, \quad B = du - \frac{u - \bar{u}}{v - \bar{v}} dv.$$

ω'_1 is Kähler homogeneous under the action of $G_1^J(\mathbb{R})$ on \mathcal{X}_1^J , $((h, \alpha), (v, u)) \rightarrow (v_1, u_1)$:

$$(2.35) \quad v_1 = h \cdot v = \frac{av + b}{cv + d}, \quad u_1 = \frac{u + nv + m}{cv + d}, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), \quad \alpha = m + in,$$

where the matrices g in (2.24) and h in (2.35) are related by (2.31).

3. FUNDAMENTAL CONJECTURE FOR THE SIEGEL-JACOBI DISK AND DOMAIN

Firstly, we fix the terminology [14]. A complex analytic manifold is *Kählerian* if it is endowed with a Hermitian metric whose imaginary part ω has $d\omega = 0$. A coset space is *homogenous Kählerian* if it carries a Kählerian structure invariant under the group. We call a *homogeneous Kähler diffeomorphism* a diffeomorphism $\phi : M \rightarrow N$ of homogeneous Kähler manifolds such that $\phi^* \omega_N = \omega_M$.

Let us remind the *fundamental conjecture for homogeneous Kähler manifolds* (Gindikin -Vinberg): *every homogenous Kähler manifold is a holomorphic fiber bundle over a homogenous bounded domain in which the fiber is the product of a flat homogenous Kähler manifold and a compact simply connected homogenous Kähler manifold.* The compact case was considered by Wang [48], Borel [14] and Matsushima [33] have considered the case of a transitive reductive group of automorphisms, while Gindikin and Vinberg [47] considered the transitive automorphism group. We mention also the essential contribution of Piatetski-Shapiro in this field [15]. The complex version, in the formulation of Dorfmeister and Nakajima [16], essentially asserts that: *every homogenous Kähler manifold, as a complex manifold, is the product of a compact simply connected homogenous manifold (generalized flag manifold), a homogenous bounded domain, and \mathbb{C}^n/Γ , where Γ denotes a discrete subgroup of translations of \mathbb{C}^n .*

Proposition 2. *Let us consider the Kähler two-form ω_1 (2.28), G_1^J -invariant invariant under the action (2.24) of G_1^J on the homogenous Kähler Siegel-Jacobi disk \mathcal{D}_1^J . We have the homogenous Kähler diffeomorphism*

$$(3.1) \quad FC : (\mathcal{D}_1^J, \omega_1) \rightarrow (\mathcal{D}_1 \times \mathbb{C}, \omega_0) = (\mathcal{D}_1, \omega_{\mathcal{D}_1}) \otimes (\mathbb{C}, \omega_{\mathbb{C}}), \quad \omega_0 = FC^*(\omega_1),$$

$$FC : z = \eta - w\bar{\eta}, \quad FC^{-1} : \eta = \frac{z + w\bar{z}}{1 - |w|^2},$$

$$(3.2) \quad \omega_0 = \omega_{\mathcal{D}_1} + \omega_{\mathbb{C}}; \quad -i\omega_{\mathcal{D}_1} = \frac{2k}{(1 - w\bar{w})^2} dw \wedge d\bar{w}, \quad -i\omega_{\mathbb{C}} = d\eta \wedge d\bar{\eta}.$$

The Kähler two-form (3.2) is invariant at the action of G_1^J on $\mathbb{C} \times \mathcal{D}_1$, $((g, \alpha), (\eta, w)) \rightarrow (\eta_1, w_1)$,

$$(3.3) \quad \eta_1 = a(\eta + \alpha) + b(\bar{\eta} + \bar{\alpha}), \quad w_1 = \frac{aw + b}{bw + \bar{a}}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(1, 1).$$

We have also the homogenous Kähler diffeomorphism

$$FC_1 : (\mathcal{X}_1^J, \omega'_1) \rightarrow (\mathcal{X} \times \mathbb{C}, \omega'_0) = (\mathcal{X}_1, \omega_{\mathcal{X}_1}) \times (\mathbb{C}, \omega_{\mathbb{C}}), \quad \omega'_0 = FC_1^*(\omega'_1),$$

$$(3.4) \quad FC_1 : 2iu = (v + i)\eta - (\bar{v} - i)\bar{\eta}; \quad FC_1^{-1} : \eta = \frac{u\bar{v} - \bar{u}v + i(\bar{u} - u)}{\bar{v} - v},$$

where ω'_1 is the Kähler two-form (2.34), $G_1^J(\mathbb{R})$ -invariant to the action (2.35), and

$$(3.5) \quad \omega'_0 = \omega_{\mathcal{X}_1} + \omega_{\mathbb{C}}, \quad \mathrm{id}\omega_{\mathcal{X}_1} = \frac{2k}{(v - \bar{v})^2} dv \wedge d\bar{v}.$$

Proof. The idea is to use the transformation (2.33) and the EZ (Eichler-Zagier) coordinates (3.6), (cf. the definition at p. 12 and p. 51 in [13] adapted to our notation)

$$(3.6) \quad v = x + iy; \quad u = pv + q, \quad x, p, q, y \in \mathbb{R}, y > 0,$$

and come back from v to w . So, let

$$z = 2i \frac{u}{v + i} = 2i \frac{pv + q}{v + i},$$

where, by the (inverse Cayley) transform (2.32), $v = -i \frac{w+1}{w-1}$. We have $z = q + ip + w(-q + ip)$, and if denote $\eta = q + ip$, where $q, p \in \mathbb{R}$, then $z = \eta - w\bar{\eta}$, with η appearing already in (2.28), and $A = d\eta - wd\bar{\eta}$. The last term in (2.28) becomes

$$(3.7) \quad \frac{A \wedge \bar{A}}{1 - |w|^2} = d\eta \wedge d\bar{\eta} = 2idp \wedge dq.$$

Vice-versa, we have $d\eta = \frac{A+w\bar{A}}{1-|w|^2}$, with A given in (2.28).

For the second assertion, we introduce the transformation (2.33) $z = 2iu(v + i)^{-1}$ in (3.1) and we get: $2i(u - \bar{u}) = (\eta - \bar{\eta})(v - \bar{v})$. Than B in (2.34) becomes

$$B = \frac{1}{2i} [(v + i)d\eta - (v - i)d\bar{\eta}]$$

and we get (3.5). ■

Corollary 1. *Let us denote $\mathcal{F} := F \circ FC$, $\mathcal{K} = K \circ FC$. In the variables (η, w) , the scalar product (2.19) becomes*

$$(3.8) \quad K(w, \eta; \bar{w}', \bar{\eta}') = (1 - w\bar{w}')^{-2k} \exp \mathcal{F}, \quad \text{where}$$

$2\mathcal{F} = 2(\bar{\eta}\zeta + |\eta'|^2) - w\bar{\eta}^2 - \bar{w}'\eta'^2 + (1 - w\bar{w}')^{-1}(-2|\zeta|^2 + w\bar{\zeta}^2 + \bar{w}'\zeta^2)$, $\zeta = \eta - \eta'$ and, for $w = w', \zeta = 0$, (2.20) becomes

$$(3.9) \quad \mathcal{K} = (1 - w\bar{w})^{-2k} \exp \mathcal{F}, \quad 2\mathcal{F} = 2\eta\bar{\eta} - \bar{w}\eta^2 - w\bar{\eta}^2.$$

Under the FC transform, the scalar product (2.25) on \mathcal{D}_1^J , with G_1^J -invariant measure $d\nu$ at the action (2.24), becomes the scalar product (3.10) on $\mathbb{C} \times \mathcal{D}_1$, with the measure $d\nu'$ (3.11), G_1^J -invariant at the action (3.3) (also $d\omega_0 \wedge d\bar{\omega}_0 = -8kd\nu'$):

$$(3.10) \quad (\phi, \psi) = \Lambda_1 \int_{\eta \in \mathbb{C}; |w| < 1} \bar{f}_\phi(\eta, w) f_\psi(\eta, w) (1 - w\bar{w})^{2k} \exp(-\mathcal{F}) d\nu', \quad \text{where}$$

$$(3.11) \quad d\nu' = \frac{d\Re w d\Im w}{(1 - w\bar{w})^2} d\Re \eta d\Im \eta.$$

3.1. Geodesics on \mathcal{D}_1^J . Now we look at the effect of the FC transform on the equations of geodesics on \mathcal{D}_1^J . We recall (cf. [5])

Remark 3. *The equations of the geodesics on the manifold \mathcal{D}_1^J , endowed with the two-form (2.28) in the variables $(w, z) \in \mathcal{D}_1 \times \mathbb{C}$, are*

$$(3.12a) \quad 2k \frac{d^2 z}{dt^2} - \bar{\eta} \left(\frac{dz}{dt} \right)^2 + 2 \left(2k \frac{\bar{w}}{P} - \bar{\eta}^2 \right) \frac{dz}{dt} \frac{dw}{dt} - \bar{\eta}^3 \left(\frac{dw}{dt} \right)^2 = 0;$$

$$(3.12b) \quad 2k \frac{d^2 w}{dt^2} + \left(\frac{dz}{dt} \right)^2 + 2\bar{\eta} \frac{dz}{dt} \frac{dw}{dt} + \left(4k \frac{\bar{w}}{P} + \bar{\eta}^2 \right) \left(\frac{dw}{dt} \right)^2 = 0,$$

where η is given by (3.1) and $P = 1 - w\bar{w}$.

If we introduce the solution

$$(3.13) \quad w = w(t) = B \frac{\tanh(t\sqrt{BB})}{\sqrt{BB}}$$

of the equations of geodesics on \mathcal{D}_1 ,

$$\frac{d^2 w}{dt^2} + 2 \frac{\bar{w}}{1 - w\bar{w}} \left(\frac{dw}{dt} \right)^2 = 0,$$

into (3.12b), then (3.12b) becomes

$$(3.14) \quad \left(\frac{dz}{dt} + \bar{\eta} \frac{dw}{dt} \right)^2 = 0.$$

Now we introduce the solution $\frac{dz}{dt} = -\bar{\eta} \frac{dw}{dt}$ of (3.14) into (3.12a) and we obtain:

$$(3.15) \quad \frac{d^2 z}{dt^2} - \frac{2\bar{w}}{P} \bar{\eta} \left(\frac{dw}{dt} \right)^2 = 0.$$

If in (3.15) we take into account (3.14), we get $\frac{d\bar{\eta}}{dt} = 0$, and a particular solution of (3.12a) consists of $(\eta = ct, w)$ with w given by (3.13). ■

4. CLASSICAL MOTION AND QUANTUM EVOLUTION

Let $M = G/H$ be a homogeneous manifold with a G -invariant Kähler two-form ω

$$(4.1) \quad \omega(z) = i \sum_{\alpha \in \Delta_+} g_{\alpha, \beta} dz_\alpha \wedge d\bar{z}_\beta, \quad g_{\alpha, \beta} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log \langle e_z, e_z \rangle.$$

Above $e_z \in \mathcal{H}$ are Perelomov's coherent state vectors, indexed by the points $z \in M$, obtained by the unitary irreducible representation π on \mathcal{H} of G , $e_z = \exp(\sum_{\alpha \in \Delta_+} z_\alpha X_\alpha)$, and Δ_+ are the positive roots of the Lie algebra \mathfrak{g} of G , with generators $X_\alpha, \alpha \in \Delta$ [37].

Passing on from the dynamical system problem in the Hilbert space \mathcal{H} to the corresponding one on M is called sometimes *dequantization*, and the dynamical system on M is a classical one [3, 4]. Following Berezin [9],[10], the motion on the classical phase space can be described by the local equations of motion $\dot{z}_\alpha = i \{ \mathcal{H}, z_\alpha \}$, $\alpha \in \Delta_+$, where \mathcal{H} is the classical Hamiltonian $\mathcal{H} = \langle e_z, e_z \rangle^{-1} \langle e_z | \mathbf{H} | e_z \rangle$ (the covariant symbol) attached to the quantum Hamiltonian \mathbf{H} , and the Poisson bracket is introduced using the matrix g^{-1} .

We consider an algebraic Hamiltonian linear in the generators \mathbf{X}_λ of the group of symmetry G

$$(4.2) \quad \mathbf{H} = \sum_{\lambda \in \Delta} \epsilon_\lambda \mathbf{X}_\lambda.$$

The classical motion generated by the Hamiltonian (4.2) is given by the equations of motion on $M = G/H$ [3, 4]:

$$(4.3) \quad i\dot{z}_\alpha = \sum_{\lambda \in \Delta} \epsilon_\lambda Q_{\lambda, \alpha}, \quad \alpha \in \Delta_+,$$

where the differential action corresponding to the operator \mathbf{X}_λ in (4.2) can be expressed in a local system of coordinates as a holomorphic first order differential operator with polynomial coefficients ($\partial_\beta = \frac{\partial}{\partial z_\beta}$),

$$(4.4) \quad \mathbb{X}_\lambda = P_\lambda + \sum_{\beta \in \Delta_+} Q_{\lambda, \beta} \partial_\beta, \quad \lambda \in \Delta.$$

We look also for the solutions of the Schrödinger equations attached to the Hamiltonian \mathbf{H} (4.2)

$$(4.5) \quad \mathbf{H}\psi = i\dot{\psi}, \quad \text{where } \psi = e^{i\varphi} \langle e_z, e_z \rangle^{-1/2} e_z.$$

We remember that [4]

Proposition 3. *On the homogenous manifold $M = G/H$ on which the holomorphic representation (4.4) is true, the classical motion and the quantum evolution generated by the linear Hamiltonian (4.2) are given by the same equation of motion (4.3). The*

phase φ in (4.5) is given by the sum $\varphi = \varphi_D + \varphi_B$ of the dynamical and Berry phase,

$$(4.6a) \quad \begin{aligned} \varphi_D &= - \int_0^t \mathcal{H}(t) dt, \quad \text{where} \\ \mathcal{H}(t) &= \sum_{\lambda \in \Delta} \epsilon_\lambda (P_\lambda + \sum_{\beta \in \Delta_+} Q_{\lambda, \beta} \partial_\beta \ln \langle e_z, e_z \rangle) \\ &= \sum_{\lambda \in \Delta} \epsilon_\lambda P_\lambda + i \sum_{\beta \in \Delta_+} \dot{z}_\beta \partial_\beta \ln \langle e_z, e_z \rangle, \end{aligned}$$

$$(4.6b) \quad \begin{aligned} \varphi_B &= -\Im \int_0^t \langle e_z, e_z \rangle^{-1} \langle e_z | d | e_z \rangle \\ &= \frac{i}{2} \int_0^t \sum_{\alpha \in \Delta_+} (\dot{z}_\alpha \partial_\alpha - \bar{\dot{z}}_\alpha \bar{\partial}_\alpha) \ln \langle e_z, e_z \rangle. \end{aligned}$$

4.1. Equations of motion on Siegel-Jacobi disk and domain. Let us consider a linear hermitian Hamiltonian in the generators of the Jacobi group G_1^J :

$$(4.7) \quad \mathbf{H} = \epsilon_a \mathbf{a} + \bar{\epsilon}_a \mathbf{a}^\dagger + \epsilon_0 \mathbf{K}_0 + \epsilon_+ \mathbf{K}_+ + \epsilon_- \mathbf{K}_-, \quad \bar{\epsilon}_+ = \epsilon_-, \quad \epsilon_0 = \bar{\epsilon}_0.$$

With Lemma 1, Proposition 3 and (2.20), we get

Proposition 4. *The equations of motion on the Siegel-Jacobi disk \mathcal{D}_1^J generated by the linear Hamiltonian (4.7) are:*

$$(4.8a) \quad i\dot{z} = \epsilon_a + \bar{\epsilon}_a w + \left(\frac{\epsilon_0}{2} + \epsilon_+ w\right) z, \quad z, w \in \mathbb{C}, \quad |w| < 1,$$

$$(4.8b) \quad i\dot{w} = \epsilon_- + \epsilon_0 w + \epsilon_+ w^2.$$

The equations of motion generated by the linear Hamiltonian (4.7) on the manifold \mathcal{X}_1^J , obtained from the equations (4.8) by partial Cayley transform (2.33) are

$$(4.9a) \quad -2\dot{v} = (\epsilon_0 + \epsilon_+ + \epsilon_-)v^2 + 2i(\epsilon_- - \epsilon_+)v + \epsilon_0 - \epsilon_- - \epsilon_+, \quad v \in \mathbb{C}, \quad \Im v > 0,$$

$$(4.9b) \quad -2\dot{u} = (\epsilon_a + \bar{\epsilon}_a)v + i(\epsilon_a - \bar{\epsilon}_a) + [(\epsilon_0 + \epsilon_+ + \epsilon_-)v + i(\epsilon_- - \epsilon_+)]u, \quad u \in \mathbb{C}.$$

For the η defined in the FC^{-1} transform (3.1), the system of first order differential equations (4.8) becomes the system of separate equations

$$(4.10a) \quad i\dot{\eta} = \epsilon_a + \epsilon_- \bar{\eta} + \frac{\epsilon_0}{2} \eta, \quad \eta \in \mathbb{C},$$

$$(4.10b) \quad i\dot{w} = \epsilon_- + \epsilon_0 w + \epsilon_+ w^2, \quad w \in \mathbb{C}, \quad |w| < 1, .$$

If in (4.9b) we make the change of variables (3.4), we get the system of decoupled equations of motion on \mathcal{X}_1^J

$$(4.11a) \quad i\dot{\eta} = \epsilon_a + \epsilon_- \bar{\eta} + \frac{\epsilon_0}{2} \eta, \quad \eta \in \mathbb{C},$$

$$(4.11b) \quad -2\dot{v} = (\epsilon_0 + \epsilon_+ + \epsilon_-)v^2 + 2i(\epsilon_- - \epsilon_+)v + \epsilon_0 - \epsilon_- - \epsilon_+, \quad v \in \mathbb{C}, \quad \Im v > 0,$$

The equation (4.8b) (the equation (4.9a)) is a *Riccati equation* on \mathcal{D}_1 (respectively, on \mathcal{X}_1). Remark that the dynamics on the Siegel ball \mathcal{D}_1 , determined by the Hamiltonian (4.7), linear in the generators of the Jacobi group G_1^J , depends **only** on the generators of

the group $SU(1, 1)$. The Riccati equation on the \mathcal{D}_1 (4.8b) appears in literature, see e.g. equation (18.2.8) in [37] in the context of quantum oscillator with variable frequency.

1.a We consider the case of *constant coefficients* of the Hamiltonian (4.7). There are two equivalent methods to integrate the Riccati equation (4.8b).

In the equation (4.8b) we put $w = -\frac{i}{\epsilon_+} \frac{\dot{a}}{a}$, and we get for a the equation $\ddot{a} + i\epsilon_0 \dot{a} - \epsilon_- \epsilon_+ a = 0$, which has the characteristic equation $(a(t) = C_{1,2} e^{i w_{1,2} t})$

$$(4.12) \quad w_{1,2}^2 + \epsilon_0 w_{1,2} + \epsilon_+ \epsilon_- = 0; w_{1,2} = \frac{-\epsilon_0 \pm \sqrt{\Delta}}{2}, \Delta = \epsilon_0^2 - 4\epsilon_+ \epsilon_-.$$

For ϵ -s constant in (4.8b), the solution of the Riccati equation is

$$(4.13) \quad w(t) = \frac{1}{\epsilon_+} \cdot \frac{w_1 C_1 e^{i w_1 t} + w_2 C_2 e^{i w_2 t}}{C_1 e^{i w_1 t} + C_2 e^{i w_2 t}} = \frac{1}{\epsilon_+} \cdot \frac{w_1 C_1 e^{\frac{i \sqrt{\Delta}}{2} t} + w_2 C_2 e^{-\frac{i \sqrt{\Delta}}{2} t}}{C_1 e^{\frac{i \sqrt{\Delta}}{2} t} + C_2 e^{-\frac{i \sqrt{\Delta}}{2} t}},$$

where $w_{1,2}$ are given in (4.12), and in order to have $w \in \mathbb{C}$, we must have $\Delta > 0$ so, if also $\epsilon_0 > 0$, then $w_{1,2} < 0$. Imposing to the solution of the Riccati equation (4.8b) the initial condition $w(0) = w_0$, it results for $f = \frac{C_1}{C_2}$ the value $f = \frac{\epsilon_+ w_0 - w_2}{w_1 - \epsilon_+ w_0}$, and we rewrite down the solution (4.13) of (4.8b) as

$$(4.14) \quad w(t, w_0) = \frac{1}{\epsilon_+} \cdot \frac{f w_1 e^{i \sqrt{\Delta} t} + w_2}{1 + f e^{i \sqrt{\Delta} t}}.$$

For the solution (4.13) of the differential equation with constant coefficients we find out

$$1 - w\bar{w} = \frac{\sqrt{\Delta}}{\epsilon_+ \epsilon_-} \frac{-w_1 |C_1|^2 + w_2 |C_2|^2}{|C_1 e^{\frac{i \sqrt{\Delta}}{2} t} + C_2 e^{-\frac{i \sqrt{\Delta}}{2} t}|^2},$$

and the condition $w(t) \in \mathcal{D}_1$ imposes the restrictions:

$$(4.15) \quad \left| \frac{C_1}{C_2} \right| > \sqrt{\frac{w_2}{w_1}} = \frac{1 + \sqrt{1 - \delta}}{\sqrt{\delta}}, \quad \epsilon_0 > 0, \quad \Delta > 0, \quad \delta = 4 \frac{\epsilon_+ \epsilon_-}{\epsilon_0^2} < 1.$$

The second method to integrate the Riccati equation is to make the substitution $w = X/Y$, and we associate to (4.8b) the linear system of first order differential equations

$$(4.16) \quad i\dot{X} = \epsilon_- Y + \epsilon_0 X, \quad i\dot{Y} = -\epsilon_+ X.$$

We eliminate X , integrate the equation in Y , and finally, we get for w the same solution (4.13).

1.b Integration of equation (4.9a).

We write (4.9a) as

$$(4.17) \quad -\dot{v} = Av^2 + Bv + C, \quad A = \frac{1}{2}(\epsilon_- + \epsilon_+ + \epsilon_0); \quad B = i(\epsilon_- - \epsilon_+); \quad C = \frac{1}{2}(\epsilon_0 - \epsilon_- - \epsilon_+),$$

where $A, B, C \in \mathbb{R}$.

We look for a solution of (4.17) as $v = \frac{1}{A} \frac{\dot{b}}{b}$ and we have $\ddot{b} + B\dot{b} + ACb = 0$, which has the solution $b(t) = C'_{1,2} e^{i v_{1,2} t}$, with $v_{1,2} = \frac{\epsilon_+ - \epsilon_- \pm \sqrt{\Delta}}{2}$, $\Delta = \epsilon_0^2 - 4\epsilon_+ \epsilon_- > 0$, The solution

of (4.17) is

$$(4.18) \quad v(t) = \frac{i}{A} \cdot \frac{v_1 C'_1 e^{iv_1 t} + v_2 C'_2 e^{iv_2 t}}{C'_1 e^{iv_1 t} + C'_2 e^{iv_2 t}} = \frac{i}{A} \cdot \frac{v_1 C'_1 e^{\frac{i\sqrt{\Delta}}{2}t} + v_2 C'_2 e^{-\frac{i\sqrt{\Delta}}{2}t}}{C'_1 e^{\frac{i\sqrt{\Delta}}{2}t} + C'_2 e^{-\frac{i\sqrt{\Delta}}{2}t}}$$

which is complex in the same case as the solution (4.13), $\Delta > 0$.

Note that the solution (4.18) of the Riccati equation (4.9a) on \mathcal{X}_1 is related to the solution (4.13) of the Riccati equation (4.8b) on \mathcal{D}_1 by the Cayley transform (2.32) if we chose the constants such that

$$(4.19) \quad \frac{C_1}{C_2} \cdot \frac{C'_2}{C'_1} = \frac{\epsilon_+ - w_2}{\epsilon_+ - w_1}.$$

2. Suppose that we know the general solution of the Riccati equation (4.8b) obtained integrating the *time-dependent* linear system (4.16). Then (4.8a) becomes

$$(4.20) \quad i\dot{z}(t) = A(t) + B(t)z, \text{ where, } A(t) = \epsilon_a + \bar{\epsilon}_a w, \quad B(t) = \frac{\epsilon_0}{2} + \epsilon_+ w.$$

Let $z(t) = CF(t)$ be the solution of the homogeneous equation $i\dot{z}(t) = B(t)z$, where $F(t) = \exp(-i \int_{t_0}^t B(t)dt)$. Then the solution of the differential equation (4.17) is given by $C(t) = C_0 - i \int_{t_0}^t \frac{A}{F} dt$.

3. Now we look at the *decoupled* system of differential equations (4.10), also in the *autonomous case*. We write down the complex numbers as $\eta = x + iy$, $\epsilon_a = a + ib$, $\epsilon_- = m + in$, $\epsilon_0/2 = p$, and (4.10a) becomes the linear system of differential equations

$$(4.21) \quad \dot{x} = nx + (p - m)y + b, \quad \dot{y} = -(m + p)x - ny - a,$$

where the solution of the characteristic equation in $e^{(\lambda t)}$:

$$(4.22) \quad \det \begin{pmatrix} n - \lambda & p - m \\ -(p + m) & -(n + \lambda) \end{pmatrix} = 0, \quad \lambda^2 = n^2 + m^2 - p^2,$$

is $\lambda = \pm \frac{i}{2} \sqrt{\Delta}$.

The solution of (4.21) in the case of constant coefficients is

$$(4.23a) \quad x(t) = \frac{q}{2\lambda} (\alpha e^{\lambda t} - \beta e^{-\lambda t}) - \frac{q}{\lambda^2}, \quad q = nb + a(m - p), \quad \alpha, \beta \in \mathbb{C},$$

$$(4.23b) \quad y(t) = \frac{q}{2\lambda} \cdot \frac{\alpha(\lambda - n)e^{\lambda t} + \beta(\lambda + n)e^{-\lambda t}}{p - m} + \frac{\frac{nq}{\lambda^2} - b}{p - m}, \quad \lambda = i \frac{\sqrt{\Delta}}{2},$$

and we find the solution $\eta(t) = x(t) + iy(t)$ of the differential equation (4.10a)

$$(4.24a) \quad \eta(t) = Me^{i\frac{\sqrt{\Delta}}{2}t} + Ne^{-i\frac{\sqrt{\Delta}}{2}t} + P, \quad \text{where}$$

$$(4.24b) \quad M = -i\frac{q\alpha}{r\sqrt{\Delta}}(\epsilon_- + w_1); \quad N = i\frac{q\beta}{r\sqrt{\Delta}}(\epsilon_- + w_2),$$

$$(4.24c) \quad \frac{\alpha}{\beta} = \frac{\epsilon_-(\epsilon_+ + w_2)}{w_2(\epsilon_- + w_1)} = \frac{w_1(\epsilon_+ + w_2)}{\epsilon_+(\epsilon_- + w_1)}, \quad \alpha = i\frac{r}{q}(\eta(t=0) - P),$$

$$(4.24d) \quad P = \frac{4\epsilon_- \bar{\epsilon}_a - 2\epsilon_0 \epsilon_a}{\Delta}, \quad r = \frac{1}{2}(\epsilon_- + \epsilon_+ - \epsilon_0),$$

$$(4.24e) \quad q = -\frac{\epsilon_0}{4}(\epsilon_a + \bar{\epsilon}_a) + \frac{1}{2}(\epsilon_a \epsilon_+ + \bar{\epsilon}_a \epsilon_-).$$

The solution of the system of differential equations (4.8) is given by $z = \eta - w\bar{\eta}$, where $\eta(t)$ has the expression given by (4.24), while the solution $w(t)$ of (4.8b) is given by (4.13).

4. Instead of the linear hermitian Hamiltonian (4.7), we could consider the *non-hermitian Hamiltonian*

$$(4.25) \quad \mathbf{H} = \epsilon_a \mathbf{a} + \epsilon_b \mathbf{a}^\dagger + \epsilon_0 \mathbf{K}_0 + \epsilon_+ \mathbf{K}_+ + \epsilon_- \mathbf{K}_-.$$

which leads to the equations of motion (4.8), where in (4.8a), the term linear in w should have as coefficient ϵ_b instead of $\bar{\epsilon}_a$. Then, with the change of variables FC given by (3.1), we get instead of (4.10a) the equation

$$(4.26a) \quad i\dot{\eta} = (1 - \bar{w}w)^{-1}(R + S\eta + T\bar{\eta}), \quad \text{where } R = \epsilon_a + (\epsilon_b - \bar{\epsilon}_a)w - \bar{\epsilon}_b \bar{w}w,$$

$$(4.26b) \quad S = \frac{\epsilon_0}{2} + (\epsilon_+ - \bar{\epsilon}_-)w - \frac{\bar{\epsilon}_0}{2}\bar{w}w, \quad T = \epsilon_- + \left(\frac{\epsilon_0 - \bar{\epsilon}_0}{2}\right)w - \bar{\epsilon}_+ \bar{w}w.$$

The equation (4.26a) get the form (4.10a) only if the Hamiltonian (4.25) becomes the hermitian Hamiltonian (4.7), i.e. $\epsilon_b = \bar{\epsilon}_a$, $\bar{\epsilon}_0 = \epsilon_0$, and $\epsilon_- = \bar{\epsilon}_+$.

4.2. **Berry phase for \mathcal{D}_1^J .** We calculate Berry phase with (4.6b), which on \mathcal{D}_1^J reads

$$\frac{2}{i}d\varphi_B = (dw\frac{\partial}{\partial w} - d\bar{w}\frac{\partial}{\partial \bar{w}} + dz\frac{\partial}{\partial z} - d\bar{z}\frac{\partial}{\partial \bar{z}})f,$$

where f is the Kähler potential (2.27).

We have

$$f_{\bar{z}} = \eta, \quad f_{\bar{w}} = \frac{\eta^2}{2} + \frac{2kw}{1 - w\bar{w}}.$$

$$\frac{2}{i}d\varphi_B = \left(\frac{\bar{\eta}^2}{2} + \frac{2k\bar{w}}{1 - w\bar{w}}\right)dw - \left(\frac{\eta^2}{2} + \frac{2kw}{1 - w\bar{w}}\right)d\bar{w} + \bar{\eta}dz - \eta d\bar{z}.$$

But $z = \eta - w\bar{\eta}$, and the Berry phase on \mathcal{D}_1^J in the variables (w, η) is

$$(4.27) \quad \frac{2}{i}d\varphi_B = \left(\frac{2k\bar{w}}{1 - w\bar{w}} - \frac{\bar{\eta}^2}{2}\right)dw + (\bar{\eta} + \bar{w}\eta)d\eta - c\bar{c}.$$

4.3. Dynamical phase. The dynamical phase is calculated with (4.6a). Firstly we calculate *the energy function* attached to the Hamiltonian (4.7):

$$(4.28) \quad \begin{aligned} \mathcal{H} = & k\epsilon_0 + \bar{\epsilon}_a z + \epsilon_+ (2kw + \frac{z^2}{2}) + [\epsilon_a + \bar{\epsilon}_a w + (\frac{\epsilon_0}{2} + \epsilon_+ w)z] \bar{\eta} \\ & + (\epsilon_- + \epsilon_0 w + \epsilon_+ w^2) (\frac{\bar{\eta}^2}{2} + \frac{2k\bar{w}}{1-w\bar{w}}). \end{aligned}$$

Now we put into evidence that the energy function attached to the hermitian Hamiltonian (4.7) is real and write it as $\mathcal{H} = \mathcal{H}_\eta + \mathcal{H}_w$, where

$$(4.29a) \quad \mathcal{H}_\eta = \bar{\epsilon}_a \eta + \epsilon_a \bar{\eta} + \frac{1}{2} (\epsilon_+ \eta^2 + \epsilon_- \bar{\eta}^2 + \epsilon_0 \eta \bar{\eta}),$$

$$(4.29b) \quad \mathcal{H}_w = k\epsilon_0 + \frac{2k}{1-w\bar{w}} (\epsilon_+ w + \epsilon_- \bar{w} + \epsilon_0 w \bar{w}).$$

Then the solutions of (4.10) are introduced into the expression of the energy function (4.29). In the case of *constant coefficients* we use the solution (4.13) ((4.24)) and we find for the function (4.29b) (respectively, (4.29a)) the *independent of time* expression

$$(4.30a) \quad \mathcal{H}_w(w(t)) = k(\epsilon_0 + 2 \frac{-w_1^2 |C_1|^2 + w_2^2 |C_2|^2}{-w_1 |C_1|^2 + w_2 |C_2|^2}),$$

$$(4.30b) \quad \mathcal{H}_\eta(\eta(t)) = \frac{2}{\Delta} \cdot (\epsilon_+ \epsilon_a^2 + \epsilon_- \bar{\epsilon}_a^2 - \epsilon_0 |\epsilon_a|^2) - \frac{q^2}{r} |\alpha|^2.$$

We look for the critical points of the energy function (4.29), i.e. the points $(w, \eta) \in \mathcal{D}_1 \times \mathbb{C}$ for which $\frac{\partial \mathcal{H}}{\partial w} = 0$, $\frac{\partial \mathcal{H}}{\partial \eta} = 0$:

$$(4.31) \quad \frac{\partial \mathcal{H}}{\partial \eta} = \bar{\epsilon}_a + \epsilon_+ \eta + \frac{\epsilon_0}{2} \bar{\eta} = 0; \quad \frac{\partial \mathcal{H}}{\partial w} = \frac{2k}{(1-w\bar{w})^2} (\epsilon_- \bar{w}^2 + \epsilon_0 \bar{w} + \epsilon_+) = 0.$$

The solution w_c of (4.31), also the solution of $\dot{w} = 0$, is $w_c = \frac{-\epsilon_0 \pm \sqrt{\Delta}}{2\epsilon_+}$. But

$$1 - w_c \bar{w}_c = \sqrt{\Delta} \frac{-\sqrt{\Delta} \pm \epsilon_0}{2\epsilon_+ \epsilon_-},$$

and in order to assure $w_c \in \mathcal{D}_1$, we have to chose the critical value $w_c = \frac{-\epsilon_0 + \sqrt{\Delta}}{2\epsilon_+}$.

The solution η_c of $\frac{\partial \mathcal{H}}{\partial \eta} = 0$, equivalent with the solution of $\dot{x} = 0$, $\dot{y} = 0$ in (4.21), is $\eta_c = 2 \frac{2\bar{\epsilon}_a \epsilon_- - \epsilon_a \epsilon_0}{\Delta}$.

We find the *Hessian function* $H(w, \eta)$ attached to the energy function (4.29) around the critical point (w_c, η_c)

$$(4.32) \quad H(w, \eta) = gw\bar{w} + \frac{\epsilon_0}{2} \eta \bar{\eta} + \epsilon_+ \eta^2 + \epsilon_- \bar{\eta}^2, \quad \text{where} \quad g = \frac{k}{2\sqrt{\Delta}} \left(\frac{4\epsilon_+ \epsilon_-}{\epsilon_0 - \sqrt{\Delta}} \right)^2 > 0$$

In general, the critical point (w_c, η_c) of the energy function (4.29) is non-degenerate, and the Hessian function (4.32) is positive definite (of index 2) if $p + 2m > 0$ (respectively, $p + 2m < 0$).

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